Exponential growth, exponential scaling, and the discontinuous Galerkin method

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FULBRIGHT

BROWN

Gronwall's lemma

If for all $t \in (0, T)$

$$u'(t) \leq \beta(t) u(t)$$

then

$$u(t) \leq u(0)\, \mathrm{e}^{\int_0^t \beta(s) \mathrm{d}s}$$

Gronwall's lemma - integral form

Let $\beta \geq 0$. If for all $t \in (0, T)$

$$u(t) \leq \alpha + \int_0^t \beta(s) u(s) \, \mathrm{d}s$$

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Abstract Gronwall's lemma

Let M be a sequentially closed topological space which has a partial ordering $\leq$. Let $A : M \to M$ such that

- $x \leq y \Rightarrow A(x) \leq A(y)$.
- A has a unique fixed point $x^\star$.
- $A^n(x) \to x^\star$ for all x.

Then

$$x \leq A(x) \;\Rightarrow\; x \leq x^\star.$$

Proof:

- $x \leq A(x)$
- $x \leq A^n(x)$
- Limit $\Rightarrow \; x \leq x^\star$

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Advection-reaction

$$\frac{\partial u}{\partial t} + a\cdot\nabla u + cu = 0.$$

Estimates of u: multiply by u, $\int_\Omega$

$$\int_\Omega \frac{\partial u}{\partial t} u \,\mathrm{d}x + \int_\Omega a\cdot\nabla u\, u \,\mathrm{d}x + \int_\Omega cu^2 \,\mathrm{d}x = 0.$$

By Green's theorem

$$\int_\Omega a\cdot\nabla u\, u \,\mathrm{d}x = \frac{1}{2}\int_{\partial\Omega} a\cdot\mathbf{n} u^2 \,\mathrm{d}x - \frac{1}{2}\int_\Omega \mathrm{div}\, a\, u^2 \,\mathrm{d}x.$$

Therefore

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \frac{1}{2}\int_{\partial\Omega} a\cdot\mathbf{n} u^2 \,\mathrm{d}x + \int_\Omega \left(c - \tfrac{1}{2}\mathrm{div}\, a\right) u^2 \,\mathrm{d}x = 0.$$

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Continuous problem

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \int_\Omega \left(c - \tfrac{1}{2}\mathrm{div}\, a\right) u^2 \,\mathrm{d}x \le 0.$$

- Elliptic case: $c - \frac{1}{2}\mathrm{div}\, a \ge 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 \le 0.$$

- Non-elliptic case: $c - \frac{1}{2}\mathrm{div}\, a \le c_0$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 \le c_0\|u\|^2.$$

Gronwall gives

$$\|u(t)\|^2 \le e^{2c_0 t}\|u(0)\|^2.$$

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Exponential scaling I.

$$\frac{\partial u}{\partial t} + a\cdot\nabla u + cu = 0.$$

Exponential scaling:

$$u(x,t) = e^{\alpha t} w(x,t).$$

This gives

New ellipticity condition:

$$\alpha + c - \tfrac{1}{2}\mathrm{div}\, a \geq 0$$

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Exponential scaling (Nävert, 1982): Choose $\mu_0 \in \mathbb{R}^d$,

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Vector field a must point in one direction!

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Exponential scaling (Ayuso, Marini, 2009): Choose $\mu : \Omega \to \mathbb{R}$,

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Lemma – Devinatz, Ellis, Friedman (1974)

$\exists\mu : a\cdot\nabla\mu \geq \gamma_0 > 0 \iff a$ possesses neither closed curves nor stationary points.

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Exponential scaling IV.

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Weak form of exponential scaling

$$\int_\Omega \frac{\partial u}{\partial t} v + a \cdot \nabla u v + c u v \, \mathrm{d}x = 0.$$

- Set $u(x,t) = e^{\mu(x,t)} \tilde{u}(x,t)$.
- Set $\hat{v}(x,t) = e^{-\mu(x,t)} v(x,t)$:

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Continuous problem

We seek $u : Q_T = \Omega \times (0,T) \to \mathbb{R}$ such that

$$\begin{aligned} \frac{\partial u}{\partial t} + a \cdot \nabla u + cu = 0 \qquad & \text{in } Q_T, \\ u = u_D \qquad & \text{on } \partial\Omega^- \times (0,T), \\ u(x,0) = u^0(x), \quad & x \in \Omega. \end{aligned}$$

- $\mathscr{T}_h$ is a triangulation of Ω.
- $S_h = \{v_h;\, v_h|_K \in P^p(K), \forall K \in \mathscr{T}_h\}$,

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DG discretization

We seek $u_h \in C^1(0,T;S_h)$ such that

$$\left(\frac{\partial u_h}{\partial t}, v_h\right) + b_h(u_h, v_h) + c_h(u_h, v_h) = l_h(v_h), \quad \forall v_h \in S_h,$$

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Reaction form

$$c_h(u, v) = \int_\Omega cuv \,\mathrm{d}x$$

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Advection form

$$b_h(u, v) = \sum_{K\in\mathscr{T}_h}\int_K (a\cdot\nabla u)v\,\mathrm{d}x - \sum_{K\in\mathscr{T}_h}\int_{\partial K^-\setminus\partial\Omega}(a\cdot\mathbf{n})[u]v\,\mathrm{d}S$$
$$- \sum_{K\in\mathscr{T}_h}\int_{\partial K^-\cap\partial\Omega}(a\cdot\mathbf{n})uv\,\mathrm{d}S,$$

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DG discretization

We seek $u_h \in C^1(0,T;S_h)$ such that

$$\left(\frac{\partial u_h}{\partial t}, v_h\right) + b_h(u_h,v_h) + c_h(u_h,v_h) = l_h(v_h), \quad \forall v_h \in S_h,$$

Right-hand side form

$$l_h(v) = -\sum_{K\in\mathscr{T}_h}\int_{\partial K^-\cap\partial\Omega}(a\cdot\mathbf{n})u_D v\,\mathrm{d}x.$$

1. Gronwall's lemma
2. Exponential scaling
3. Discontinuous Galerkin
4. Error analysis
5. Construction of μ

Error analysis

- Error $e_h(t) := u(t) - u_h(t) = \eta_h(t_n) + \xi_h(t)$.
- Interpolation error $\eta_h(t) = u(t) - \Pi_h u(t)$.
- Unknown discrete error $\xi_h(t) = \Pi_h u(t) - u_h(t) \in S_h$.
- Error equation: For all $v_h \in S_h$

$$\left(\frac{\partial e_h}{\partial t}, v_h\right) + b_h(e_h, v_h) + c_h(e_h, v_h) = 0$$

$$\left(\frac{\partial \xi}{\partial t}, v_h\right) + \left(\frac{\partial \eta}{\partial t}, v_h\right) + b_h(\xi, v_h) + b_h(\eta, v_h) + c_h(\xi, v_h) + c_h(\eta, v_h) = 0.$$

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Error equation, $\phi = e^{-\mu}\tilde{\xi}$

$$\left(\frac{\partial \xi}{\partial t}, \Pi_h\phi\right) + b_h(\xi,\phi) + b_h(\xi, \Pi_h\phi - \phi) + b_h(\eta,\phi) + b_h(\eta, \Pi_h\phi - \phi)$$
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Assumptions on μ

$$0 \le \mu(x,t) \le \mu_{\max},$$
$$|\mu(x,t) - \mu(y,t)| \le L_\mu |x-y|,$$

Lemma

Let $\phi = e^{-\mu}\tilde{\xi}$, then

$$\|\Pi_h\phi(t) - \phi(t)\|_{L^2(K)} \le Ch\|\tilde{\xi}(t)\|_{L^2(K)},$$
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$$\left(\frac{\partial \xi}{\partial t}, \Pi_h \phi\right) = \left(\frac{\partial \xi}{\partial t}, \phi\right) = \left(e^{\mu}\frac{\partial \tilde{\xi}}{\partial t} + e^{\mu}\frac{\partial \mu}{\partial t}\tilde{\xi}, e^{-\mu}\tilde{\xi}\right)$$
$$= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{\xi}\|^2 + \left(\frac{\partial \mu}{\partial t}\tilde{\xi}, \tilde{\xi}\right).$$

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Lemma: Let $\frac{\partial\mu}{\partial t} + a\cdot\nabla\mu + c - \frac{1}{2}\mathrm{div}\, a \geq \gamma_0 > 0$, then

$$\left(\frac{\partial \xi}{\partial t}, \Pi_h \phi\right) + b_h(\xi,\phi) + c_h(\xi,\phi) \geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{\xi}\|^2 + \gamma_0\|\tilde{\xi}\|^2 + \frac{1}{2}\sum_{K\in\mathscr{T}_h}\left(\|[\tilde{\xi}]\|^2_{a,\partial K^-\setminus\partial\Omega} + \|\tilde{\xi}\|^2_{a,\partial K\cap\partial\Omega}\right).$$

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$$\int_\Omega c\xi(\Pi_h\phi - \phi)\,\mathrm{d}x \le C\|\tilde{\xi}\|h\|\tilde{\xi}\| = Ch\|\tilde{\xi}\|^2.$$

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Altogether

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{\xi}(t)\|^2 + 2\gamma_0\|\tilde{\xi}(t)\|^2 \leq Ch\|\tilde{\xi}(t)\|^2 + Ch^{2p+1}\left(|u(t)|^2_{H^{p+1}} + |u_t(t)|^2_{H^{p+1}}\right).$$

Lemma

$$\|\tilde{\xi}(t)\|^2 + \gamma_0 \int_0^t \|\tilde{\xi}(\vartheta)\|^2 \mathrm{d}\vartheta \le Ch^{2p+1}\big(|u^0|^2_{H^{p+1}} + |u|^2_{L^2(0,t;H^{p+1})} + |u_t|^2_{L^2(0,t;H^{p+1})}\big).$$

$$\|\xi(t)\| = \|e^{\mu(t)}\tilde{\xi}(t)\| \le e^{\mu_{\max}}\|\tilde{\xi}(t)\|.$$

Theorem

Let $0 \le \mu \le \mu_{\max}$, $\mu(t) \in W^{1,\infty}(\Omega)$ and $\mu_t + a\cdot\nabla\mu + c - \frac{1}{2}\mathrm{div}\, a \ge \gamma_0$ on Q_T, where $\gamma_0 > 0$. Then

$$\max_{t\in[0,T]} \|e_h(t)\| + \sqrt{\gamma_0}\|e_h\|_{L^2(Q_T)} \le Ch^{p+1/2}e^{\mu_{\max}}\big(|u^0|_{H^{p+1}} + |u|_{L^2(0,t;H^{p+1})} + |u_t|_{L^2(0,t;H^{p+1})}\big).$$

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- Find μ_1 such that
$$\frac{\partial \mu_1}{\partial t} + a\cdot\nabla\mu_1 = 1.$$
- Pathlines: $S(\cdot; x_0, t_0)$ is the trajectory of a massless particle
$$S(t_0; x_0, t_0) = x_0 \in \overline{\Omega}, \quad \frac{\mathrm{d}S(t; x_0, t_0)}{\mathrm{d}t} = a(S(t; x_0, t_0), t).$$
- Then
$$\frac{\mathrm{d}\,\mu_1(S(t; x_0, t_0), t)}{\mathrm{d}t} = \Big(\frac{\partial \mu_1}{\partial t} + a\cdot\nabla\mu_1\Big)(S(t; x_0, t_0), t) = 1,$$
therefore
$$\mu_1(S(t; x_0, t_0), t) = t - t_0.$$

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- Interpretation: $\mu_1(x,t)$ is the time a particle carried by the flow field a passing through (x,t) has spent in Ω.
- Assumption: Let the time spent in Ω by any particle carried by the flow be bounded by $\widehat{T}$. Then $0 \le \mu \le \mu_{\max}$.
- If we choose

$$\mu(x,t) = \mu_1(x,t)\Big(\Big|\inf_{Q_T}(c - \tfrac{1}{2}\mathrm{div}\, a)^-\Big| + \gamma_0\Big),$$

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- Interpretation: $\mu_1(x,t)$ is the time a particle carried by the flow field a passing through (x,t) has spent in Ω.
- Assumption: Let the time spent in Ω by any particle carried by the flow be bounded by $\widehat{T}$. Then $0 \le \mu \le \mu_{\max}$.
- If we choose

$$\mu(x,t) = \mu_1(x,t)\Big(\Big|\inf_{Q_T}(c - \tfrac{1}{2}\mathrm{div}\,a)^-\Big| + \gamma_0\Big),$$

then

$$\frac{\partial \mu}{\partial t} + a\cdot\nabla\mu + c - \tfrac{1}{2}\mathrm{div}\,a \ge \gamma_0.$$

Lemma

Let $a \in L^\infty(Q_T)$ be continuous w.r.t. time and Lipschitz continuous w.r.t. space. Let there exist $a_{\min} > 0$ such that

$$-a(x,t) \cdot \mathbf{n} \geq a_{\min}$$

for all $x \in \partial\Omega^-, t \in [0,T)$. Let the life-time of particles carried by the flow be uniformly bounded by $\widehat{T}$. Then μ is Lipschitz continuous and uniformly bounded.

Resulting estimate

$$\|e_h\|_{L^\infty(L^2)} + \sqrt{\gamma_0}\|e_h\|_{L^2(L^2)} \leq Ce^{c\widehat{T}} h^{p+1/2}.$$

Standard Gronwall estimates

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Conclusions:

- Gronwall's inequality gives exponential growth w.r.t. time.
- General form of exponential scaling.
- Error estimates for DG exponentially growing not in time but in the time particles carried by the flow spend in Ω.
- Effectively, we apply Gronwall's inequality not in the Eulerian framework, but in the Lagrangian (along individual pathlines).
- Nonlinear case possible (linear case + Zhang-Shu technique).

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Thank you for your attention